2. Complete Contracts I: Static Bilateral Contracting

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2.2 Moral Hazard II: Multitasks

2.2.1 Multitasks

For example: the little monk, teachers, CEOs, workers, salesmen, government officials, etc.
The point: the tradeoff involves incentive balance.

2.2.2 Setting

The following is from Holmstrom-Milgrom (1991). [①] Under asymmetrical information, the agent who undertakes multitasks makes a one-time choice of a vector of efforts $a=(a_1......a_n)$ with cost $c(a)$, which brings expected gross benefits $B(a)$ for the principal and generates a vector of information signal $x_i=a_i+\varepsilon_i$ for every task, where $\varepsilon \sim (0,\Sigma)$. Σ is a $n\times n$ matrix. Wage is $w(x)=\alpha+\beta_1x_1+\beta_2x_2+...+\beta_kx_k=\alpha+\beta^T x$ [②], and $u(w)=-\exp(-r\hat{w})$. The risk-neutral P's problem is

$$\underset{\{\alpha,\beta,a\}}{Max}\, EV[B(a)-w(x)]$$

$$s.t.\quad \text{(IR)}\quad Eu(w)-c(a)\geq E(\underline{U})$$

$$\text{(IC)}\quad a\in\arg\max_a Eu(\hat{w})$$

$$EV=B(a)-\alpha-\beta^T a\,;$$

IR: ACE=$\alpha+\beta^T a-c(a)-\frac{1}{2}r\beta^T\Sigma\beta\geq\underline{U}$; IC: $\beta^T=\frac{\partial c(a)}{\partial a_i}=c_i(a)$, $\frac{\partial\beta^T}{\partial a}=[c_{ij}]$.

$\therefore \alpha=\underline{U}-\beta^T a+\frac{1}{2}r\beta^T\Sigma\beta+c(a)$. Substituting α into EV (by SPE solution),

① This is the most cited paper by the two authors. As of March 31, 2026, Google Scholar has over 9769 citations.
② Mathematically it could be βx^T, or β=(β1->β2->…->βn), such that β^T=(β1, β2,…,βn).

equivalently we have

$$TCE = \underset{\beta,a}{Max}\, B(a) - \frac{1}{2} r\beta^T \Sigma\beta - c(a)$$

Differentiating with respect to a, we have

$B'(a) - r\Sigma\beta \frac{\partial\beta^T}{\partial a} - c_i(a) = 0$. Because $\frac{\partial\beta^T}{\partial a} = c_{ij}$, and $c_i(a) = \beta^T$, we get

$$B'(a) = \beta(r\Sigma[c_{ij}] + I)$$

$$\beta = \frac{B'(a)}{I + r\Sigma[c_{ij}]} \quad (3\text{-}1)$$

[Note] If a and x are both one dimension variable, i.e. $x = a + \varepsilon$, $c(a) = \frac{ba^2}{2}$, expression (3-1) reduces to $\beta = \frac{1}{1 + rb\sigma^2}$.

2.2.3 Case I: The effort costs are independent

As a benchmark, when the error terms are stochastically independent (Σ is diagonal matrix) and the activities technically independent ($c_{ij} = 0$ for $i \neq j$), expression (3-1) is simplified to

$$\beta = \frac{B'(a)}{1 + rc_{ii}\sigma^2}$$

In this case, commissions are independent each other, so the logic is the same as the single-task model. When $\sigma^2 \to +\infty$，$\beta = 0$；$\sigma^2 \to 0$，$\beta = 1$．$r\uparrow$，$\beta\downarrow$．$c_{ii}\uparrow$，$\beta\downarrow$．

2.2.4 Case II: The effort costs are not independent

In the general case, $c_{ij} \neq 0$. Specifically, $a = (a_1, a_2)$, only x_1 is observable, and $x_1 = a_1 + \varepsilon$, $\sigma_2^2 = +\infty$, $\sigma_{12} = 0$ (independent outcome). Now according to (3-1), $(I + r[c_{ij}]\Sigma)$ becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 + rc_{11}\sigma_1^2 & rc_{12}\sigma_2^2 \\ rc_{21}\sigma_1^2 & 1 + rc_{22}\sigma_2^2 \end{bmatrix},$$

$$\therefore\ \beta = \frac{B'(a)}{I + r\Sigma[c_{ij}]}$$

$$= \frac{1}{(1+rc_{11}\sigma_1^2)(1+rc_{22}\sigma_2^2) - r^2 c_{12}c_{21}\sigma_1^2\sigma_2^2} \begin{bmatrix} 1+rc_{22}\sigma_2^2 & -rc_{12}\sigma_2^2 \\ -rc_{21}\sigma_1^2 & 1+rc_{11}\sigma_1^2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(1+rc_{22}\sigma_2^2)B_1 - rc_{12}\sigma_2^2 B_2}{(1+rc_{11}\sigma_1^2)(1+rc_{22}\sigma_2^2) - r^2 c_{12}c_{21}\sigma_1^2\sigma_2^2} \\ \dfrac{-rc_{21}\sigma_1^2 B_1 + (1+rc_{11}\sigma_2^2)B_2}{(1+rc_{11}\sigma_1^2)(1+rc_{22}\sigma_2^2) - r^2 c_{12}c_{21}\sigma_1^2\sigma_2^2} \end{bmatrix}$$

$$\therefore \beta_1 = \frac{(1+rc_{22}\sigma_2^2)B_1 - rc_{12}\sigma_2^2 B_2}{(1+rc_{11}\sigma_1^2)(1+rc_{22}\sigma_2^2) - r^2 c_{12}c_{21}\sigma_1^2\sigma_2^2}$$

$$= \frac{B_1 + (rc_{22}B_1 - rc_{12}B_2)\sigma_2^2}{1+rc_{11}\sigma_1^2 + (rc_{22} + r^2 c_{11}c_{22}\sigma_1^2 - r^2 c_{12}c_{21}\sigma_1^2)\sigma_2^2}$$

When $\sigma_2^2 \to \infty$, according to L' Hopital Rule, we have

$$\beta_1 = \frac{rc_{22}B_1 - rc_{12}B_2}{rc_{22} + r^2 c_{11}c_{22}\sigma_1^2 - r^2 c_{12}c_{21}\sigma_1^2} = \frac{B_1 - B_2\, c_{12}/c_{22}}{1+r\sigma_1^2(c_{11} - c_{12}^2/c_{22})} \tag{3-2}$$

The sign of c_{12} is key. When $c_{12} < 0$, that is to say two tasks are complementary in A's cost function, $|c_{12}|\uparrow$, $\beta_1\uparrow$; otherwise, $c_{12} > 0$, that is to say two tasks are substitutes, $c_{12}\uparrow$, $\beta_1\downarrow$.

If $c_{12} < 0$ and $\sigma_1^2 \to 0$, $\beta_1 \to 1$; if $c_{12} > 0$ and $\sigma_1^2 > 0$, β_1 is intermediate; if $\sigma_1^2 \to +\infty$, $\beta_1 \to 0$, i.e. fixed wage is the optimal.

2.2.5 Applications

(1) Missing incentive clauses in contracts

A little explicit incentive provisions are incorporated into contracts, especially for home construction contracts which absent timely completion terms.[①] Because P cannot monitor *other aspects* of A's performances. Suppose that a_1 is unmeasurable ($\sigma_1^2 = \infty$), a_2 is measurable ($\sigma_2^2 = \sigma^2$), and $x = \mu(a_1, a_2) + \varepsilon$. $B(0, a_2) = 0$ for all $a_2 \geq 0$, i.e. a_1 is very important.

[①] Economist Grossman in the field of contract theory included time and other conditions in the contract with the construction contractor when building his house, but forgot to include "the roof cannot leak" in the contract. As a result, he lost the lawsuit after discovering the leak and suing the contractor.

a. $\beta = 0$. Suppose $a_1 + a_2 = \bar{a} = Cont$ and $c'(\bar{a}) = 0$ (e.g., $c(a) = \frac{(a-\bar{a})^2}{2}$), i.e. A will spend minimal positive efforts and even take pleasure in some limit. $\therefore TCE = B(a_1, \bar{a} - a_1) - \frac{1}{2} r\beta^2\sigma^2 - c(\bar{a}) = B(a_1, \bar{a} - a_1) - c(\bar{a})$.

b. $\beta > 0$. $a_1 = 0$, $\therefore TCE = B(0, \hat{a}) - \frac{1}{2} r\beta^2\sigma^2 - c(\hat{a}) = -\frac{1}{2} r\beta^2\sigma^2 - c(\hat{a}) < 0$.

c. $\beta < 0$. $a_2 = 0$, $a_1 < \bar{a}$, because $c'(a_1) < 0 = c'(\bar{a})$. Then we have $TCE = B(a_1, 0) - \frac{1}{2} r\beta^2\sigma^2 - c(a_1) < B(\bar{a}, 0) - c(\bar{a}) < B(a_1, \bar{a} - a_1) - c(\bar{a})$ (When we apply convex preference assumption).

(2) Low-powered incentives in firms

Williamson (1975, 1985) pointed out that the incentives offered to employees who use employer's assets are generally "low-powered" compared to the "high-powered" incentives offered to independent contractors who have own assets. Suppose that expected gross profits include two parts: expected net revenue $B(a_1)$, and the expected change in net asset value $V(a_2)$. $B(0) = V(0) = 0$, and a_2 is unobservable, so $x = B(a_1) + \varepsilon_x$.

According to previous analysis, we know that P should set $\beta = 0$. That is to say, he pays the employee fixed wage, which is so-called "low-powered" incentives in firms. Fox example, the wage of salesmen for new products is often fixed, but not for old products. And so does contract system for plants in China, $\because TCE_e = B(a_1) + V(\bar{a} - a_1) - c(\bar{a})$.

(3) Asset ownership allocation

Contrary to *employment*, under *contracting* (or sale contract) the expected change in net asset value accrues to independent contractor (the owner of assets). In order to incentivize A (the contractor) to spend efforts on revenues, assume that $\beta > 0$, and $a_1 + a_2 > \bar{a}$. Let A's optimal efforts are (a_1^*, a_2^*). $\therefore TCE_c = B(a_1^*) + V(a_2^*) - \frac{1}{2} r(\beta^2\sigma_x^2 + \sigma_v^2) - c(a_1^* + a_2^*)$.

If $TCE_e \geq TCE_c$, employment dominates contracting; vice versa.

Optimal ownership allocation trades off between incentives and insurance. Example: taxi ownership.

(4) Limits on outside activities

A common feature of employment contracts is that employer has authority to restrict employee's outside activities during business hours, and sometimes after hours as well (for example, official). Employees in "responsible positions" are allowed to more freedom of action than other employees. Why?

We begin with the assumption that it is easier for an employer to exclude an activity entirely than to monitor it and limits its extent. A has a finite pool $K=\{1,\ldots K\}$ of potential activities. P allows A to engage only in a subset of tasks $A\subset K$. To simplify, let $A=\{a\}$ and perfect measurable. a_k is A's personal business and unmeasurable. A's net cost is

$$c(a_1,\ldots,a_N)=c(a+\sum\nolimits_k a_k)-\sum\nolimits_k v_k(a_k)$$

Where, v_k is return from A's personal activities. When $k\notin A$, $a_k=0$, so we could replace Σ_k with Σ_A. $B(a_1,\ldots,a_N)=pa$, and $x(a_1,\ldots,a_N)=a+\varepsilon$.

P's problem is to determine commission rate β and subset A. At the first step, we fix β and consider $A(\beta)$. A's problem is to maximize his ACE respect to a_i

$$\alpha+\beta a+\sum\nolimits_A v_k(a_k)-\frac{1}{2}r\beta^2\sigma^2-c(a+\sum\nolimits_A a_k)$$

FOC:

$$\beta=c'(a+\sum\nolimits_A a_k) \tag{3-3}$$

$$v_k'(a_k)=c'(a+\sum\nolimits_A a_k) \tag{3-4}$$

$$\therefore \beta=v_k'(a_k) \tag{3-5}$$

Expression (3-5) indicates that efforts that spend on outside activities are independent of A. That is to say, given β, if permitted, A will allocate all the efforts to tasks away from P's interest. The benefit of allowing A to spend efforts on task k is $v_k(a_k(\beta))$, while the (opportunity) cost is pa_k. p is marginal product in the principal's task. Therefore, by reallocating fix payoff α [①], the optimal set of allowable personal tasks is

$$A(\beta)=\{k\in K\mid v_k(a_k(\beta))>pa_k\}$$

Note that the higher is A's marginal reward for performance in the main job, the greater is his freedom to pursue personal business.

① Coase Theorem (1960).

At the first step, given A, then P determines β:

$$\underset{\beta}{Max}(pa+\sum\nolimits_{A(\beta)}v_k(a_k)-\frac{1}{2}r\beta^2\sigma^2-c(a+\sum\nolimits_{A(\beta)}a_k)) \tag{3-6}$$

s.t. (3-3), (3-4)

Solution:

According to (3-3) and (3-4), we get

$$\frac{\partial a}{\partial \beta}=\frac{1}{c''(\cdot)} \tag{3-7}$$

Differentiating expression (3-6) with respect to β, we have

$$p\frac{\partial a}{\partial \beta}+\sum\nolimits_{A(\beta)}v_k'(a_k)\frac{\partial a_k}{\partial \beta}-r\sigma^2\beta-(\frac{\partial a}{\partial \beta}+\sum\nolimits_{A(\beta)}\frac{\partial a_k}{\partial \beta})c'(\cdot)=0$$

Invert (3-7) into (3-6), we have

$$p\frac{1}{c''(\cdot)}+\sum\nolimits_{A(\beta)}v_k'(a_k)\frac{\partial a_k}{\partial \beta}-r\sigma^2\beta-(\frac{1}{c''(\cdot)}+\sum\nolimits_{A(\beta)}\frac{\partial a_k}{\partial \beta})c'(\cdot)=0$$

$\because\ v_k'(a_k)=c'(a+\sum\nolimits_{A(\beta)}a_k)=\beta$,

$$\therefore\ p\cdot\frac{1}{c''(\cdot)}-r\sigma^2\beta-\frac{1}{c''(\cdot)}\beta=0$$

$$\therefore\ \beta=\frac{p}{1+rc''\sigma^2} \tag{3-8}$$

Expression (3-8) tells that if it becomes easier to measure A's performance ($\sigma^2\downarrow$), or A become less risk averse ($r\downarrow$), then A's marginal reward $\beta\uparrow$, and his personal business activities will be less curtailed ($A\uparrow$). In the limit, $\sigma^2=0$, $\beta=p$, $A=K$.

These give an explanation to the freedom of CEO and officials in government.

(5) Allocating tasks among two agents

There are two jobs ($k=1,2$) and two identical agents (i,j). Assume that $c(a_{1i}+a_{2i})=\frac{(a_{1i}+a_{2i})^2}{2}$, $B=B(a_{1i}+a_{1j})+B(a_{2i}+a_{2j})$, $x_1=(a_{1i}+a_{1j})+\varepsilon_1$, $x_2=(a_{2i}+a_{2j})+\varepsilon_2$.

If two agents are jointly responsible, i.e. every agent does two jobs, $w_i = \alpha_i + \beta_{1i}x_1 + \beta_{2i}x_2$. Agent i's CE is

$$CE_i = \alpha_i + \beta_{1i}(a_{1i} + a_{1j}) + \beta_{2i}(a_{2i} + a_{2j}) - \frac{1}{2}r\beta_{1i}^2\sigma_1^2 - \frac{1}{2}r\beta_{2i}^2\sigma_2^2 - c(a_{1i} + a_{2i}) \quad (3\text{-}9)$$

Expression (3-9) is differentiated with respect to a_{1i} and a_{2i}, we have

(IC1) $\beta_{1i} = c'(a_{1i} + a_{2i}) = a_{1i} + a_{2i}$, $\beta_{2i} = c'(a_{1i} + a_{2i}) = a_{1i} + a_{2i}$ (3-10)

Likely, we also have

(IC2) $\beta_{1j} = c'(a_{1j} + a_{2j}) = a_{1j} + a_{2j}$ and $\beta_{2j} = c'(a_{1j} + a_{2j}) = a_{1j} + a_{2j}$ (3-11)

So, in order to assure $a_{1i} > 0$ and $a_{2i} > 0$, A must have $\beta_{1i} = \beta_{2i} = \beta$, or $\beta_i = \beta_j \equiv \beta$. It is so-called "equal compensation principle" (Milgrom-Roberts, 1991). So, we have

$$TCE^{joint} = B(a_{1i} + a_{1j}) + B(a_{2i} + a_{2j}) - \frac{1}{2}r\beta^2(\sigma_1^2 + \sigma_2^2) - c(a_{1i} + a_{2i}) - c(a_{1j} + a_{2j})$$

P's problem is to maximize TCE^{joint} subjected to express (3-10) and (3-11), that is

$$\underset{\beta}{Max}\, 2[B(\beta) - c(\beta) - \frac{1}{2}r\beta^2(\sigma_1^2 + \sigma_2^2)] \quad (3\text{-}12)$$

FOC:

$$\beta = \frac{B'}{1 + r(\sigma_1^2 + \sigma_2^2)} \quad (3\text{-}13)$$

If every agent is solely responsible, for example agent i and j does work 1 and 2, respectively. We have

$$CE_1 = \alpha_1 + \beta_{1i}(a_{1i}) - \frac{1}{2}r\beta_{1i}^2\sigma_1^2 - c(a_{1i})$$

(IC3) $\beta_{1i} = c'(a_{1i}) = a_{1i}$ (3-14)

Likely,

(IC4) $\beta_{2j} = c'(a_{2j}) = a_{2j}$ (3-15)

$$\begin{aligned} TCE^{sole} &= B(a_{1i}) + B(a_{2j}) - \frac{1}{2}r\beta_{1i}^2\sigma_1^2 - \frac{1}{2}r\beta_{2j}^2\sigma_2^2 - c(a_{1i}) - c(a_{2j}) \\ &= B(\beta_{1i}) - \frac{1}{2}r\beta_{1i}^2\sigma_1^2 - c(\beta_{1i}) + B(\beta_{2j}) - \frac{1}{2}r\beta_{2j}^2\sigma_2^2 - c(\beta_{2j}) \end{aligned} \quad (3\text{-}16)$$

P's problem is to maximize TCE^{sole} subjected to express (3-13) and (3-14), that is

$$\beta_{1i} = \frac{B'}{1 + r\sigma_1^2} \text{ and } \beta_{2j} = \frac{B'}{1 + r\sigma_2^2}.$$

Obviously $\beta < \beta_{1i}$ and $\beta < \beta_{2j}$. By envelope theorem, (3-12) is smaller than (3-16), so $TCE^{joint} < TCE^{sole}$. Division of labor is better than not, because the agents face less risk ($\sigma^2 < \sigma_1^2 + \sigma_2^2$).

Multitask model has been extended to more complicated situation. Sometimes task separation is the optimal in the presence of direct conflicts between tasks (Dewatripont and Tirole, 1999). And the separation of ex ante and ex post monitoring can then help avoid cover ups (Dewatripont and Tirole, 1994). Hong et. al. (2018) test the theory of multitasking using evidence from a natural field experiment in Chinese factories.

Another application in China: 聂辉华，2006，《取消农业税对乡镇政府行为的影响：一个多任务委托代理模型》，**《世界经济》**，第 8 期

2.2.6 Firm as an incentive system *

The theory of the firm[1]

TCE: Coase (1937), Cheung (1983), Willamson (1975, 1985, 1986)--- design of job
PRT: Grossman-Hart (1986), Hart-Moore (1990), Hart (1995)-- property rights
P-A: Alchian-Demsetz (1972), Holmstrom (1982)-- wage

Holmstrom-Milgrom (1994)

All above theories deal with make-or-buy decision, or answer the question that what's the nature of the firm, but can not answer why does inside procurement tend to involve production by a worker who is supervised by the firm *and* uses the firm's tools *and* is paid a fixed wage? Why does outside procurement tend to involve purchases from a worker who chooses his or her own methods and hours *and* owns the tools used *and* is paid only for quantities supplied?

That is: (1) fixed wage + employer's tools + employer supervision
(2) commission rates + contractor's tools + worker freedom

The problem is how to integrate three instruments—wage, ownership, and design of job and why make choice between different incentive systems? The intuitive idea is that increasing the incentive for just one task could cause a worker to devote too much effort to that one task while neglecting other aspects of the job, and that increase incentives for all of the agent's activities avoids that cost. Just recall multitask agency model (Holmstrom-Milgrom, 1991). Logic:

Exogenous Parameters	Cost of measuring performance	⇒	Performance rewards
	Asset specificity		Return form ownership
	Uncertainty		Worker freedom

[1] A best reference is Holmstrom, Bengt and Jean Tirole, 1989, "The Theory of the Firm", in *Handbook of Industrial Organization*, R. Willig (eds.), Amsterdam: North Holland.

A general model is to design such variables: commission rates β, allocation of transferable property-right returns λ, exclusion of private returns δ and monitoring intensity Σ.

$$\pi_P = B(a) - \alpha - \sum_i \beta_i x_i + \sum_j (1-\lambda_j) Y_j - K(\Sigma)$$

$$\pi_A = \alpha + \sum_i \beta_i x_i + \sum_j \lambda_j Y_j + \sum_k \delta_k Z_k - c(a)$$

$$ACE(a,\beta,\lambda,\delta,\Sigma) = \alpha + \sum_i \beta_i a + \sum_j \lambda_j G_j(a) + \sum_k \delta_k H_k(a) - c(a) - \frac{1}{2} rV(\beta,\lambda,\delta,\Sigma)$$

$$\max[TCE(a,\beta,\lambda,\delta,\Sigma)]$$
$$\equiv B(a) + \sum_j \lambda_j G_j(a) + \sum_k \delta_k H_k(a) - c(a) - \frac{1}{2} rV(\beta,\lambda,\delta,\Sigma) - K(\Sigma)$$

$$s.t.\quad a = \arg\max_{\hat{a}} ACE$$

A *supermodular* function: $f(z \vee z') + f(z \wedge z') \geq f(z) + f(z')$, i.e., $\frac{\partial^2 f}{\partial z_i \partial z_j} \geq 0$. Or $f(1,1) + f(0,0) \geq f(1,0) + f(0,1)$.

Let $f(x,\pi)$ be supermodular, where x is an *n*-vector of endogenous variables and π an *m*-vector of parameters.

Another application in China: 吕冰洋、聂辉华，2014，《弹性分成：分税制的契约与影响》，《经济理论与经济管理》，第 7 期。

On supermodular definition: 周鹏，2007，《互补性与超模性经济理论述评》，《经济学动态》，第 12 期。

Think: Why works get piece-rate wage and restrict freedom without ownership in almost factories in south China?

2.2.7 Summary

- When A burdens multitasks, what matter is to coordinate different tasks.
- When efforts are complementary among multitasks, the incentive problem is simple.
- When efforts are substitutable among multitasks, increasing incentive in one task will decrease incentive in other tasks.
- When efforts are substitutable and some tasks are unobservable, P should restrict employee’s outside activities.