3. Complete Contracts II: Static Multilateral Contracting

Huihua NIE
School of Economics
Renmin University of China
www.niehuihua.com
niehuihua(at)vip.163.com

3.3 Adverse Selection V: Auction

3.3.1 Introduction

Auction is the most important pricing mechanism. In fact, in terms of "market design", auction theory is one of the fastest fields in microeconomics in recently decades. Paul Milgrom and Robert Wilson won the Nobel Prize in auction theory in 2020.[1] Generally we can classify auctions on what is being auctioned. In a **private-value auction**, a bidder can learn nothing about his value from knowing the values of the other bidders. An example is the sale of antique chairs to people who will not resell them. Furthermore, we distinguish between the **independent private-value auction**, in which knowing his own value tells him nothing about other bidders' values, and other situations such as the **affiliated private-value auction** in which he might be able to use knowledge of his own value to deduce something about other players' values. In a **pure common-value auction**, the bidders have identical values, but each bidder forms his own estimate on the basis of his own private information. An example is bidding for U. S. Treasury bills. Actually private-value and common-value auctions both can be regarded as a special case of affiliated-value auction.

There are at least four widely known and used auctions with independent values—the English auction, the Dutch auction, the first-price sealed-bid auction and the second-price sealed-bid auction.

English auction It is the best-known auction, also called ascending price auction. The good is sold to the highest bidder at the price offered by that bidder, provided that the highest bid exceeds the introductory price set by the seller.

Dutch auction It is also called descending price auction. The good is sold to the first buyer to stop the process at his price.[2]

First-price sealed-bid auction All the buyers simultaneously offer a price in a sealed envelope, and the good is sold at the highest price offer. If two or more bidders offer the highest price, then the good is allocated randomly to one of then at the highest price.

Second-price sealed-bid auction All the rules are the same as first-price sealed-bid auction, except that the good is sold at the second-highest price offer. It is also named as Vickrey auction.

See the Chinese movie named 《大腕》, the 58th minute and the end.

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[1] They ever established a company with website address which is "marketdesign.com", but now it has been closed.
[2] I give some examples in my book 《一切皆契约》.

3.3.2 Revenue equivalence theorem

If the seller and buyers are risk neutral and if the buyers' valuations are independently drawn from the same distribution, it turns out that it does not matter which of these auctions the seller chooses: they all yield the same expected revenue. Furthermore, the English auction and the Vickrey auction yield the same symmetric equilibrium outcome in this case. The Dutch and first-price sealed-bid auctions are strategically equivalent.

Suppose a seller faces two buyers with i.i.d. valuations:

$$v_i = \begin{cases} v_H & with \quad probability \quad \beta \\ v_L & with \quad probability \quad 1-\beta \end{cases}$$

Consider first that the English and Vickrey auctions, where each bidder has a unique (weakly) **dominant strategy**: In the English auction the dominant strategy is to "continue" until the ascending asking price process hits the reservation value of the bidder and then to "stop". In the Vickrey auction each bidder's dominant strategy is to offer a price equal to his reservation value. Proof omitted. It is the same as Groves-Clark mechanism.

Theorem: In the 2nd price auction, the equilibrium outcome is efficient.

Proof: The buyer who has the highest valuation wins the good. Details omitted.

Since the equilibrium prices and the final allocations in the unique symmetric dominant-strategy equilibrium are the same under the English and Vickrey auctions, the seller ought to be indifferent between them. In fact, the seller's expected revenue is

$$\beta^2 v_H + 2\beta(1-\beta)v_L + (1-\beta)^2 v_L = \beta^2 v_H + (1-\beta^2)v_L$$

The equilibrium probabilities of winning the auction for a low- and high-type buyer are respectively, $\frac{1-\beta}{2}$ and $(1-\beta)+\frac{\beta}{2}$. And the expected payoff of the buyer with v_L is 0, while a v_H buyer gets an expected payoff of $(1-\beta)(v_H - v_L)$.

Consider now the Dutch and first-price sealed-bid auctions. The information set (no available pricing information of others), strategy sets and the allocation rules are the same in both auctions. Notice first that there is no dominant strategy. It is then less clear what the outcome of the game is likely to be for the latter two auctions, but the symmetric Bayesian equilibrium is the most plausible outcome for these games.

The reason is the following: if the bidder 1 chooses the pure strategy $b_1(v) = v$, then the bidder 2's best response is to set $b_2(v_L) = b_1(v_L) = v_L$ and $b_2(v_H) = b_1(v_L) + \varepsilon$. But bidder 1's best response is to set $b_1(v_H) = v_L + 2\varepsilon$, and so on. Because a v_H buyer doesn't know which type another buyer is, so he must mix his strategies such that he must be different between low price v_L and high price $\bar{b}$ that makes him winning with probability 1, which makes

$$(1-\beta)(v_H - v_L) = 1\times(v_H - \bar{b}) \Rightarrow \bar{b} = \beta v_H + (1-\beta)v_L$$

where $\bar{b} \in (v_L, v_H)$.

A v_L buyer's equilibrium strategy is simply the pure strategy $b(v_L) = v_L$, because the competition between two low-type buyers is Bertrand price-competition.

By symmetry, the equilibrium probabilities of winning the auction for a low- and high-type buyer are respectively, $\frac{1-\beta}{2}$ and $(1-\beta)+\frac{\beta}{2}$, which are the same as in English and Vickrey auctions, so are their expected payoff (0 for low-type and $(1-\beta)(v_H - v_L)$ for high-type). And given that, for each type, the bidders' equilibrium expected payoffs and probabilities of winning the auction are the same in all four auctions, the seller's expected revenue must also be the same in all four auctions.

Maskin-Riley (1989) proves the equivalent theorem with a continuum of types. Klemperer (1999) provides a formal statement as follows:

Revenue Equivalence Theorem: Assume each of n risk-neutral potential buyers has a privately-known value independently drawn from a common distribution $F(v)$ that is strictly increasing and atomless on $(\underline{v}, \bar{v})$. Suppose that no buyer wants more than one of the *k* available identical indivisible objects. Then any auction mechanism in which (i) the objects always go to the k buyers with the highest values, and (ii) any bidder with value $\underline{v}$ expects zero surplus, yields the same expected revenue, and results in a buyer with value v making the same expected payment.

BD (2005, 7.3.6) shows that the buyers' bidding behavior in the English and Vickrey auctions are unaffected by their attitudes toward risk. However, the Dutch and first-price sealed-bid auctions generate higher expected revenue for the seller than the English and Vickrey auctions when buyers are risk averse.

Think

Since that all the four auctions are equivalent, why usually we use the English auction?

English auction dominates in conditioning atmosphere, transparence. But all the auctions, especially Vickrey auction, will suffer from collusion.

3.3.3 Winner's curse

Now let's consider common value auction. Suppose that an item has a value $v \in \{H, L\}$, where $H > L > 0$. Nobody knows the true value of the item. The seller and two buyers have a prior belief $\frac{1}{2}$ that the item has a high value. The buyers receive an independent private estimate of the value or signal, $s_i \in \{s_H, s_L\}$, and thus have private information about the value of the item. When the value of the item is H, the probability of receiving signal s_H is $p > \frac{1}{2}$. Similarly, when the value is L, the probability of receiving signal s_L is $p > \frac{1}{2}$. Thus a buyer's *prior* expected value on receiving the signal s_H is given by $v_H = E[v \mid s_H] = pH + (1-p)L$. Similarly, $v_L = E[v \mid s_L] = pL + (1-p)H$.

Consider only a Vickrey auction by revenue equivalence theorem.

Case I: one buyer gets the signal s_H and the other the signal s_L. The winner with signal s_H will get the item at the price v_L, but his *posterior* expected value conditional on winning-- v_w --is

$$v_w = E[v \mid (s_H, s_L)] = \frac{L+H}{2} < pH + (1-p)L = v_H$$

Proof:

$$\begin{aligned} v_w &= E[v \mid (s_H, s_L)] \\ &= H \times P[H \mid (s_H, s_L)] + L \times P[L \mid (s_H, s_L)] \\ &= H \times \frac{P[(s_H, s_L) \mid H]P(H)}{P(s_H, s_L)} + L \times \frac{P[(s_H, s_L) \mid L]P(L)}{P(s_H, s_L)} \end{aligned}$$

$$\begin{aligned} P(s_H, s_L) &= P\left[\left(s_H, s_L\right) \middle| H\right]P(H) + P\left[\left(s_H, s_L\right) \middle| L\right]P(L) \\ &= \frac{1}{2}p(1-p) + \frac{1}{2}(1-p)p \\ &= p(1-p) \end{aligned}$$

$$\begin{aligned} \therefore v_w &= H \times \frac{P[(s_H, s_L) \mid H]P(H)}{P(s_H, s_L)} + L \times \frac{P[(s_H, s_L) \mid]P(L)}{P(s_H, s_L)} \\ &= \frac{1}{2}\frac{Hp(1-p)}{p(1-p)} + \frac{1}{2}\frac{L(1-p)p}{p(1-p)} \\ &= \frac{H+L}{2} \end{aligned}$$

It means that the winner has to revise downward his expect value, but he still makes a positive net gain, since

$$v_w = E[v \mid (s_H, s_L)] = \frac{L+H}{2} > pL + (1-p)H = v_L$$

Case II: both buyers get the signal s_H, each one has a 50% chance of winning and paying the price v_H.

$$v_w = E[v \mid (s_H, s_H)] = \frac{p^2 H + (1-p)^2 L}{p^2 + (1-p)^2} > pH + (1-p)L = v_H$$

It means that the winner will revise upward expected value ex post, so he gets positive net payoff. There is a "**winner's blessing**".

Case III: both buyers get a signal s_L, each one has a 50% chance of winning and paying the price v_L.

$$v_w = E[v \mid (s_L, s_L)] = \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2} < pL + (1-p)H = v_L$$

It means that the winner has to revise downward his expect value, and his net gain is negative. There is "**winner's curse**".

But the previous strategies are *not* equilibrium strategies. Here is the Bayesian equilibrium: when the buyers receive the signal s_L, they bid $b_L = E[v \mid (s_L, s_L)] = \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2}$; and $b_H = E[v \mid (s_H, s_H)] = \frac{p^2 H + (1-p)^2 L}{p^2 + (1-p)^2}$ for s_H. Then there is no winner's curse or winner's blessing. If one buyer receives the signal s_H but another receives s_L (his optimal biding is still $b_L < v_L$), the former bids $b = b_H$ is weakly dominant strategy. Hence the expected revenue from the Vickrey auction is

$$\frac{1}{2}[p^2 - (1-p)^2](H-L) + \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2}.$$

Proof:

If both buyers get signal s_H with $P(s_H, s_H) = \frac{1}{2}[p^2 + (1-p)^2]$, the seller will receive b_H, otherwise he receives b_L, so the seller's expected revenue is:

$$EV = \frac{p^2 + (1-p)^2}{2} b_H + [1 - \frac{p^2 + (1-p)^2}{2}] b_L$$
$$= \frac{[p^2 - (1-p)^2]}{2}(H - L) + \frac{p^2 L + (1-p)^2 H}{p^2 + (1-p)^2}$$

3.3.4 Extension

(1) Auction of 3G license in Europe: succeed for English, but failed in Dutch. The point is the number of licenses should be less than that of incumbents. English auction firstly, and sealed auction secondly.

(2) Regulation and auction/bid. Regulation for medicines' profit rate leads to the higher price in the process of bid for hospitals.

References

Klemperer, Paul, 2004, *Auctions Theory and Practice*, New Jersey: Princeton University Press; http://www.nuff.ox.ac.uk/users/klemperer/VirtualBook/VirtualBookCoverSheet.asp.